Robertson-type Theorems for Countable Groups of Unitary Operators

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ABSTRACT. Let $\mathcal{G}$ be a countably infinite group of unitary operators on a complex separable Hilbert space H. Let $X = \{x_1, ..., x_r\}$ and $Y = \{y_1, ..., y_s\}$ be finite subsets of H, $r < s$, $V_0 = \overline{span}\,\mathcal{G}(X)$, $V_1 = \overline{span}\,\mathcal{G}(Y)$ and $V_0 \subset V_1$. We prove the following result: Let W_0 be a closed linear subspace of V_1 such that $V_0 \oplus W_0 = V_1$ (i.e., $V_0 + W_0 = V_1$ and $V_0 \cap W_0 = \{0\}$). Suppose that $\mathcal{G}(X)$ and $\mathcal{G}(Y)$ are Riesz bases for V_0 and V_1 respectively. Then there exists a subset $\Gamma = \{z_1, ..., z_{s-r}\}$ of W_0 such that $\mathcal{G}(\Gamma)$ is a Riesz basis for W_0 if and only if $g(W_0) \subseteq W_0$ for every g in $\mathcal{G}$. We first handle the case where the group is abelian and then use a cancellation theorem of Dixmier to adapt this to the non-abelian case. Corresponding results for the frame case and the biorthogonal case are also obtained.

1. Introduction

We first consider wavelet-type problems associated with countably infinite abelian groups of unitary operators on a complex separable Hilbert space. Other results in such and similar settings can be found in [4, 9, 10, 17]. We then adapt this to non-abelian groups using a cancellation theorem of Dixmier.

Section 2 is a revisit of Robertson's theorems [12] in the setting of countably infinite abelian groups of unitary operators. Recently, Han, Larson, Papadakis and Stavropoulos [10] extended Robertson's original results to this setting, with proofs involving the spectral theorem for an abelian group of unitary operators and certain non-trivial facts on von Neumann algebras. Using some simple observations on harmonic analysis, we show how Robertson's original elementary proofs in [12] still work in this new setting. In section 3, we consider the problem of existence of oblique multiwavelets, and show that oblique multiwavelets exist under a very natural assumption. The results here extend those of [14, 16]. Finally in section 4, we discuss the non-abelian case using a cancellation theorem of Dixmier.

Let us set up some notations and terminologies. Throughout this paper, let H denote a complex separable Hilbert space. The inner product of two vectors x and

y in H is denoted by $\langle x, y \rangle$. A countable indexed family $\{v_n\}_{n \in J}$ of vectors in H is a *Riesz basis* for its closed linear span $V = \overline{span}\{v_n\}_{n \in J}$ if there exist positive constants A and B such that

$$(1.1) \qquad A \sum |a_n|^2 \le \| \sum a_n v_n \|^2 \le B \sum |a_n|^2, \quad \forall \{a_n\} \in \ell^2(J).$$

A countable indexed family $\{v_n\}_{n \in J}$ is a *frame* for its closed linear span V if there exist positive constants A and B such that

$$(1.2) \qquad A\|f\|^2 \le \sum |\langle f, v_n \rangle|^2 \le B\|f\|^2, \quad \forall f \in V.$$

It is well known that a Riesz basis for a Hilbert space is a frame for the same space. Two families $\{v_n\}$ and $\{\tilde{v}_n\}$ in H are *biorthogonal* if

$$(1.3) \qquad \langle v_n, \tilde{v}_m \rangle = \delta_{n,m} \quad \forall n, m.$$

If V and W are closed linear subspaces of H such that $V \cap W = \{0\}$ and the vector sum $V_1 = V + W$ is closed, then we write $V_1 = V \oplus W$ and call this a *direct sum*. In this case, the map $P : V_1 \longrightarrow V_1$ defined by

$$(1.4) \qquad P(v + w) = v, \quad v \in V, w \in W,$$

is called the (oblique) projection of V_1 on V along W. For the special case when V and W are orthogonal, we shall write $V_1 = V \oplus^\perp W$ and call this an *orthogonal direct sum*. We write $V^\perp$ for the orthogonal complement of V in H.

Let G be a discrete group. For every g in G, let χ_g denote the characteristic function of $\{g\}$. Then $\{\chi_g : g \in G\}$ is an orthonormal basis for $\ell^2(G)$. For each g in G, define $l_g : \ell^2(G) \longrightarrow \ell^2(G)$ by $(l_g a)(h) = a(g^{-1}h)$, $h \in G$. Then $l_g(\chi_h) = \chi_{gh}$ for all g, h in G. The left regular representation λ of G is the homomorphism $\lambda : g \mapsto l_g$.

2. Robertson's Theorem for Countable Abelian Groups of Unitary Operators

Let $B(H)$ be the space of all bounded linear maps on H. A unitary system $\mathcal{U}$ in $B(H)$ is a set of unitary operators on H which contains the identity operator on H. A closed linear subspace M of H is a *wandering subspace* for $\mathcal{U}$ if $U(M) \perp V(M)$ for all U, V in $\mathcal{U}$ with $U \ne V$. It was first observed in [7] that Robertson's results on wandering subspaces for the cyclic group generated by a single unitary operator [12] have interesting connection to the existence of orthonormal wavelets associated with orthonormal multiresolutions. Subsequently, it was noted in [8] that analogues of Robertson's results hold for the group generated by a finite set of commuting unitary operators. Recently, Han, Larson, Papadakis and Stavropoulos extended Robertson's results to the setting of a countable abelian group of unitary operators [10, Theorem 4]. Their proof uses the spectral theorem for an abelian group of unitary operators and certain non-trivial facts on von Neumann algebras. The purpose of this section is to show, with the help of some simple observations, how Robertson's original elementary proof in [12] can easily be carried over, almost verbatim, to this new setting.

LEMMA 2.1. *Let M and K be wandering subspaces of H for a countable group $\mathcal{G}$ of unitary operators on H. If $\sum_{g \in \mathcal{G}} \oplus^\perp g(M) \subseteq \sum_{g \in \mathcal{G}} \oplus^\perp g(K)$, then $dim(M) \le dim(K)$.*

PROOF. We modify the arguments due to I. Halperin, given in [**6**, p. 17], for the case of a single unitary operator. The assertion is trivial if $dim(K) = \infty$. Therefore suppose that $dim(K) = k < \infty$. Let $\{x_1, ..., x_k\}$ be an orthonormal basis for K. Then $\{gx_j : g \in \mathcal{G}, j = 1, ..., k\}$ is an orthonormal basis for $K_1 = \sum_{g \in \mathcal{G}} \oplus^{\perp} g(K)$. Let $\{y_i\}_{i \in I}$ be an orthonormal basis for M, so $\{gy_i : g \in \mathcal{G}, i \in I\}$ is orthonormal in K_1. By Bessel's inequality,

$$\sum_{g \in \mathcal{G}} \sum_{i \in I} |\langle x_j, gy_i \rangle|^2 \leq \|x_j\|^2 = 1, \quad j = 1, ..., k.$$

Hence

$$\begin{aligned}
dim(M) &= \sum_{i \in I} \|y_i\|^2 \\
&= \sum_{i \in I} \sum_{g \in \mathcal{G}} \sum_{j=1}^{k} |\langle y_i, gx_j \rangle|^2 \\
&= \sum_{j=1}^{k} \sum_{g \in \mathcal{G}} \sum_{i \in I} |\langle x_j, g^{-1}y_i \rangle|^2 \\
&\leq k.
\end{aligned}$$

$\square$

Let us recall some results on harmonic analysis (see, for example, [**13**]). Let G be a locally compact abelian group, and $\hat{G} = \{\gamma : G \longrightarrow \mathbb{T} \text{ continuous characters}\}$ the dual group of G.

Fact 1: G is isomorphic and homeomorphic to $\hat{\hat{G}}$, the dual group of $\hat{G}$, via the canonical map $g \longrightarrow e_g$, where $e_g(\gamma) = \gamma(g)$, $\gamma \in \hat{G}$.

Fact 2: The Fourier transform on $L^1(G)$ is the map $\wedge : L^1(G) \longrightarrow C_0(\hat{G})$ defined by

$$\hat{f}(\gamma) = \int_G f(g)\overline{\gamma(g)} \, dg, \quad \gamma \in \hat{G}, \quad f \in L^1(G).$$

Identifying $\hat{\hat{G}}$ with G as in Fact 1, the Fourier transform on $L^1(\hat{G})$ is the map $\wedge : L^1(\hat{G}) \longrightarrow C_0(\hat{\hat{G}}) = C_0(G)$ defined by

$$\hat{f}(g) = \int_{\hat{G}} f(\gamma)\overline{\gamma(g)} \, d\gamma, \quad g \in G, \quad f \in L^1(\hat{G}).$$

Fact 3: If G is compact abelian, then $\hat{G}$ forms an orthonormal basis for $L^2(G)$. Correspondingly, if G is discrete abelian, then $\hat{\hat{G}} = \{e_g : g \in G\}$ forms an orthonormal basis for $L^2(\hat{G})$.

Fact 4: Suppose that G is a discrete abelian group (so $\hat{G}$ is compact and $L^2(\hat{G}) \subset L^1(\hat{G})$). If f is in $L^2(\hat{G})$, then $\hat{f}(g) = \langle f, e_g \rangle$, $g \in G$, and $\hat{f}$ is in $\ell^2(G)$. Moreover,

(i)   the Fourier transform $\wedge : L^2(\hat{G}) \longrightarrow \ell^2(G)$ is a unitary operator;

(ii) $\hat{e}_g = \chi_g$ for every g in G;

(iii)   for all f, h in $L^2(\hat{G})$, we have $\widehat{fh} = \hat{f} * \hat{h}$, i.e.,

$$\widehat{fh}(g) = \sum_{m \in G} \hat{f}(m)\hat{h}(gm^{-1}), \quad g \in G.$$

(iv)   for every g in G, we have $\wedge \circ M_g = l_g \circ \wedge$, where $M_g : L^2(\hat{G}) \longrightarrow L^2(\hat{G})$ is the multiplication operator defined by

$$(M_g f)(\gamma) = \gamma(g) f(\gamma), \quad \gamma \in \hat{G}.$$

Using Fact 4 (i)–(iii), the proofs of Theorem 1 and Theorem 2 in [**12**] can now be carried over verbatim to our new setting, with $\mathbb{Z}$ and $[0, 2\pi)$ there replaced by $\mathcal{G}$ and $\widehat{\mathcal{G}}$ respectively. Hence we have

THEOREM 2.2. *([**10**, Theorem 4]) Let $\mathcal{G}$ be a countably infinite abelian group of unitary operators on H, let X and Y be wandering subspaces of H for $\mathcal{G}$ such that*

(a) $\sum_{g \in \mathcal{G}} \oplus^{\perp} g(X) \subseteq \sum_{g \in \mathcal{G}} \oplus^{\perp} g(Y)$,
(b) $dim(Y) < \infty$.

Then there exists a wandering subspace X' of H for $\mathcal{G}$ such that

(i) $g(X) \perp h(X')$ *for all* $g, h \in \mathcal{G}$,
(ii) $\sum_{g \in \mathcal{G}} \oplus^{\perp} g(X) \oplus^{\perp} \sum_{g \in \mathcal{G}} \oplus^{\perp} g(X') = \sum_{g \in \mathcal{G}} \oplus^{\perp} g(Y)$,
(iii) $dim(X) + dim(X') = dim(Y)$.

Related results for the setting of frames can be found in [**9**] and [**17**].

3. Oblique Multiwavelets and Biorthogonal Multiwavelets

Throughout this section, let $\mathcal{G}$ be a countably infinite abelian group of unitary operators on a complex separable Hilbert space H. Let $X = \{x_1, ..., x_r\}$ and $Y = \{y_1, ..., y_s\}$ be finite subsets of H, $r < s$, $V_0 = \overline{span}\,\mathcal{G}(X)$, $V_1 = \overline{span}\,\mathcal{G}(Y)$ and

$$(3.1) \qquad\qquad\qquad\qquad V_0 \subset V_1.$$

THEOREM 3.1. *Let W_0 be a closed linear subspace of V_1 such that $V_0 \oplus W_0 = V_1$. Suppose that $\mathcal{G}(X)$ and $\mathcal{G}(Y)$ are Riesz bases for V_0 and V_1 respectively. Then there exists a subset $\Gamma = \{z_1, ..., z_{s-r}\}$ of W_0 such that $\mathcal{G}(\Gamma)$ is a Riesz basis for W_0 if and only if*

$$(3.2) \qquad\qquad\qquad g(W_0) \subseteq W_0, \quad g \in \mathcal{G}.$$

Such types of oblique multiwavelets were previously studied in [**1, 2, 15, 16**]. For the proof of Theorem 3.1, we need the following elementary result from [**11**, Lemma 3.5].

LEMMA 3.2. *Let M, M' and N be linear subspaces of a vector space X such that*

$$X = M \oplus N = M' \oplus N.$$

Let P be the projection of X on M along N and let Q be the projection of X on M' along N. Then $P_1 = P|_{M'} : M' \longrightarrow M$ and $Q_1 = Q|_M : M \longrightarrow M'$ are invertible, and $P_1^{-1} = Q_1$.

Proof of Theorem 3.1. The "only if" part is obvious. Conversely, suppose that (3.2) holds. Let $V = V_1 \cap V_0^{\perp}$, the orthogonal complement of V_0 in V_1. Then we have

$$V_1 = V_0 \oplus^{\perp} V = V_0 \oplus W_0.$$

By [**11**, Proposition 2.2] and Theorem 2.2, there exists $Z = \{w_1, ..., w_{s-r}\} \subset V$ such that $\mathcal{G}(Z)$ is an orthonormal basis for V. Let $P : V_1 \longrightarrow V_1$ be the oblique

projection of V_1 on W_0 along V_0. By Lemma 3.2, $P|_V : V \longrightarrow W_0$ is invertible. Hence $\{Pgw_j : g \in \mathcal{G}, \ j = 1, ..., s-r\}$ is a Riesz basis for W_0. Since $g(V_0) = V_0$ for every $g \in \mathcal{G}$, and (3.2) holds, P commutes with $g|_{V_1}$ for every $g \in \mathcal{G}$. Therefore, $\mathcal{G}(\Gamma)$ is a Riesz basis for W_0, where $z_j = Pw_j$, $j = 1, ..., s-r$, and $\Gamma = \{z_1, ..., z_{s-r}\}$. $\square$

A corresponding result for frames is also valid.

THEOREM 3.3. *Let W_0 be a closed linear subspace of V_1 such that $V_0 \oplus W_0 = V_1$ (i.e., $V_0 + W_0 = V_1$ and $V_0 \cap W_0 = \{0\}$). Suppose that $\mathcal{G}(X)$ and $\mathcal{G}(Y)$ are frames for V_0 and V_1 respectively. Then there exists a subset $\Gamma = \{z_1, ..., z_p\}$ of W_0 such that $\mathcal{G}(\Gamma)$ is a frame for W_0 if and only if*

$$(3.3) \qquad g(W_0) \subseteq W_0, \quad g \in \mathcal{G}.$$

Note that here the number p is not necessarily equal to $s - r$, as in the corresponding case for Riesz bases. We omit the proof of Theorem 3.3, which is similar to that of Theorem 3.1. Instead of applying Theorem 2.2, we use the following simple observation (see [**9**]): if $\{v_n\}_{n \in J}$ is a frame for V_1 and $P^\perp$ is the orthogonal projection of V_1 onto $V = V_1 \cap V_0^\perp$, then $\{P^\perp(v_n)\}_{n \in J}$ is a frame for V.

As a corollary of the oblique case in Theorem 3.1, we obtain the following result for the biorthogonal case, which is an extension of [**14**, Theorem 3.6].

COROLLARY 3.4. *Let $X = \{x_1, ..., x_r\}$, $\tilde{X} = \{\tilde{x}_1, ..., \tilde{x}_r\}$, $Y = \{y_1, ..., y_s\}$ and $\tilde{Y} = \{\tilde{y}_1, ..., \tilde{y}_s\}$ be finite subsets of H. Let $\mathcal{G}(X), \mathcal{G}(\tilde{X}), \mathcal{G}(Y)$ and $\mathcal{G}(\tilde{Y})$ be Riesz bases for their closed linear spans $V_0, \tilde{V}_0, V_1$ and $\tilde{V}_1$ respectively, $\mathcal{G}(X)$ biorthogonal to $\mathcal{G}(\tilde{X})$, $\mathcal{G}(Y)$ biorthogonal to $\mathcal{G}(\tilde{Y})$, and*

$$V_0 \subset V_1, \quad \tilde{V}_0 \subset \tilde{V}_1.$$

Let $W_0 = V_1 \cap \tilde{V}_0^\perp$ and $\tilde{W}_0 = \tilde{V}_1 \cap V_0^\perp$. If $r < s$, then

(i) *there exists a subset $\Gamma := \{z_1, ..., z_{s-r}\}$ of W_0 such that $\mathcal{G}(\Gamma)$ is a Riesz basis for W_0 and $\mathcal{G}(X \cup \Gamma)$ is a Riesz basis for V_1, and*

(ii) *there exists a subset $\tilde{\Gamma} := \{\tilde{z}_1, ..., \tilde{z}_{s-r}\}$ of $\tilde{W}_0$ such that $\mathcal{G}(\tilde{\Gamma})$ is a Riesz basis for $\tilde{W}_0$ and $\mathcal{G}(\tilde{X} \cup \tilde{\Gamma})$ is a Riesz basis for $\tilde{V}_1$, and $\mathcal{G}(\tilde{\Gamma})$ is biorthogonal to $\mathcal{G}(\Gamma)$.*

PROOF. For every $g \in \mathcal{G}$, we have $g(V_j) = V_j$ and $g(\tilde{V}_j) = \tilde{V}_j$ for $j = 0, 1$. As g is unitary, using the definitions of W_0 and $\tilde{W}_0$, $g(W_0) = W_0$ and $g(\tilde{W}_0) = \tilde{W}_0$ too. Since $V_1 = V_0 \oplus W_0$, by Theorem 3.1, there exists a subset $\Gamma := \{z_1, ..., z_{s-r}\}$ of W_0 such that $\mathcal{G}(\Gamma)$ is a Riesz basis for W_0. By [**14**, Proposition 3.1], $\tilde{W}_0 \oplus W_0^\perp = H$. Hence by [**16**, Lemma 3.1], there exists a subset $\tilde{\Gamma} := \{\tilde{z}_1, ..., \tilde{z}_{s-r}\}$ of $\tilde{W}_0$ such that $\mathcal{G}(\tilde{\Gamma})$ is a Riesz basis for $\tilde{W}_0$ and $\mathcal{G}(\tilde{\Gamma})$ is biorthogonal to $\mathcal{G}(\Gamma)$. The remaining assertions follow easily from [**14**, Theoerm 2.1]. $\square$

4. Non-Abelian Groups

We consider now the possibility that $\mathcal{G}$ is not abelian. The proofs of the major statements all follow from the following fundamental result:

THEOREM 4.1 (Cancellation Theorem). *Let $\mathcal{G}$ be a locally compact group, and let ρ be a unitary representation of $\mathcal{G}$ whose commutant is a finite von Neumann*

algebra. Suppose ρ is equivalent to $\sigma_1 \oplus \sigma_2$ as well as $\sigma_1 \oplus \sigma_3$. Then σ_2 is equivalent to σ_3.

The proof of this theorem is an application of a result due to Dixmier [**5**, III.2.3 Proposition 6]. We will use the Cancellation Theorem in the following specific case, altered slightly from [**3**].

PROPOSITION 4.2. *Suppose $\mathcal{G}$ is a discrete group, and ρ is a representation of $\mathcal{G}$ which is equivalent to a finite multiple of the left regular representation of $\mathcal{G}$. Then the commutant of ρ is a finite von Neumann algebra, whence the subrepresentations of ρ satisfy the cancellation property. That is to say, if ρ is equivalent to $\sigma_1 \oplus \sigma_2$ as well as $\sigma_1 \oplus \sigma_3$, then σ_2 is equivalent to σ_3.*

THEOREM 4.3. *The statements in Theorems 2.2, 3.1, 3.3, and Corollary 3.4 still hold when the assumption that $\mathcal{G}$ is abelian is removed.*

PROOF. We prove Theorem 2.2, without the assumption that $\mathcal{G}$ is abelian. The other statements follow analogously.

Let $K = \sum_{g \in \mathcal{G}} \oplus^{\perp} g(Y)$ and let ρ denote the action of $\mathcal{G}$ on K. Since Y is a complete wandering subspace of K for $\mathcal{G}$ and is finite dimensional, we have that ρ is equivalent to a finite multiple of the left regular representation λ of $\mathcal{G}$, whence we can apply the cancellation theorem. Let $K_1 = \sum_{g \in \mathcal{G}} \oplus^{\perp} g(X)$; let σ_1 denote the action of $\mathcal{G}$ on K_1, and let σ_2 denote the action of $\mathcal{G}$ on $K \cap K_1^{\perp}$, the orthogonal complement of K_1 in K.

For every positive integer N, let λ_N denote the N multiple of the left regular representation λ of $\mathcal{G}$. We have the following equivalences:

$$\lambda_{dim(X)} \oplus \lambda_{dim(Y)-dim(X)} \simeq \lambda_{dim(Y)} \simeq \rho \simeq \sigma_1 \oplus \sigma_2$$

as well as

$$\sigma_1 \simeq \lambda_{dim(X)},$$

since X is a complete wandering subspace for σ_1 on K_1.

Therefore we have that

$$\sigma_1 \oplus \lambda_{dim(Y)-dim(X)} \simeq \sigma_1 \oplus \sigma_2.$$

By the cancellation theorem, we have that

$$\sigma_2 \simeq \lambda_{dim(Y)-dim(X)},$$

from which it follows that $K \cap K_1^{\perp}$ contains a complete wandering subspace of dimension $dim(Y) - dim(X)$; call this subspace X'. The items (i) (ii) and (iii) from Theorem 2.2 now follow. □

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